

Standard Model gauge couplings unification

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Abstract

We study the low energy evolution of the Standard Model gauge coupling constants. Using a special normalization of the group generators, we show that the three low energy gauge couplings can be determined in terms of one independent parameter.

1 Introduction

The Standard Model (SM) is a mathematically consistent renormalizable field theory which predicts or is consistent with all experimental facts. It successfully predicted the existence and form of the weak neutral current, the existence and masses of the W and Z bosons, and the charm quark, as necessitated by the GIM mechanism. The charged current weak interactions, as described by the generalized Fermi theory, were successfully incorporated, as was quantum electrodynamics. The consistency between theory and experiment indirectly tested the radiative corrections and ideas of renormalization and allowed the successful prediction of the top quark mass. Nevertheless, despite the apparent striking success of the theory, there are a lot of reasons why it is not the ultimate theory. First there is the well-established experimental observations of neutrino oscillations which are impossible in the SM. Secondly, some values of the SM parameters are not calculable in

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the theory, notably, the fermion mass hierarchy, the hierarchy of symmetry-breaking scales, and the Higgs boson mass. Hence the theory has far too much arbitrariness to be the final story. Finally, there exist purely theoretical difficulties in describing hadrons by means of the available methods of quantum field theory. These and other deficiencies of the SM motivated the effort to construct theories with higher unification of gauge symmetries.

In the framework of the Grand Unification hypothesis, it is possible to obtain a reasonable explanation of the relation $\Lambda_{QCD} \ll M_{GUT}$ that is based on the logarithmic renormalization-group dependence of the gauge coupling constant on the energy. One immediate consequence of the Grand Unification hypothesis is a very simple explanation for the experimentally observed charge quantization. This is because the eigenvalues of the generators of a simple non-Abelian group are discrete while those corresponding to the Abelian group are continuous. Unfortunately, by now LEP data has shown that simple non-SUSY grand unifications must be excluded, initially by the increased accuracy in the measurement of the Weinberg angle, and by early bounds on the proton lifetime. Nevertheless, the Grand Unification hypothesis remains very attractive, especially as the grand unifications group does not necessarily be simple. For example, such group may be represented as a product of identical simple groups (with the same coupling constants by some discrete symmetries) or it may be obtained as an extension of a simple Lie group by means of a finite group of operators. The latter possibility will be discussed in the paper. In Sec 2, we study the extended $Spin(8)$ symmetry and construct the symmetry breaking of the group down to subgroups of the SM group. In Sec 3, we study the evolution of the SM low-energy gauge couplings. We show that the three SM gauge couplings can be determined in terms of one independent parameter. The final section is devoted to discussions and comments.

2 $S_3 \ltimes Spin(8)$ symmetry

We begin by discussing the following simple construction. This construction arises in connection with the following question: is it possible to embed an arbitrary group G in some group \tilde{G} with the property that every automorphism of G is the restriction of some inner automorphism of \tilde{G} ? Let Φ be a subgroup of $\text{Aut } G$. Then for \tilde{G} one may take the set of ordered pair ϕg with

multiplication defined by the rule

$$\phi g \cdot \phi' g' = \phi \phi' g^{\phi'} g', \quad (1)$$

there $\phi \in \Phi$ and $g \in G$. (We are writing pairs without their customary comma and brackets.) The group axioms are straightforward to verify. From the rule for multiplication (1) it is immediate that $\phi^{-1} g \phi = g^\phi$. Hence the problem is solved. The group \tilde{G} is called the extension of the group G by means of the automorphisms in Φ and denoted as $\Phi \ltimes G$. Alternatively one says that \tilde{G} is a semidirect product of G by Φ .

Now let Φ be a subgroup of the outer automorphisms group of G . Suppose V is a representation space of G . Then the representation of G in V induces a representation of \tilde{G} in the direct sum $\tilde{V} = V_1 \oplus \cdots \oplus V_n$, where each direct summand V_i is isomorphic to V and $n = |\Phi|$. For the space V_i one may take the ordered pair $\phi_i V$, where $\phi_i \in \Phi$. Then the action of \tilde{G} on \tilde{V} can be written as

$$\phi g \cdot \oplus_i \phi_i V = \oplus_i \phi \phi_i g^{\phi_i} V. \quad (2)$$

Let G_0 be a set of elements of G such that $g^\phi = g$ for all $\phi \in \Phi$. Clearly, it is a Φ -invariant subgroup of G . Using the formula (2) one may define a representation of G_0 on \tilde{V} . It is easy to prove that G_0 acts equivalently on each direct summand of \tilde{V} .

Let G be a simple gauge group. If the normalization of the generators of G are fixed, then the gauge couplings will be the same for both G and G_0 . Suppose that for the energy scale $\mu > M_0$ an G gauge theory possesses both discrete and gauge symmetries, whereas for $\mu = M_0$ the symmetries breaking

$$\tilde{G} \rightarrow G \rightarrow G_0 \quad (3)$$

to take place. Then the representation space of G_0 is reduced to V and hence the normalization of the generators of G_0 are changed. Since the definition of coupling constants depends on the normalization of the generators it follows that the gauge coupling of G_0 should be also change, namely $g^2 \rightarrow g^2 |\Phi|$ as $\tilde{V} \rightarrow V$.

Now we suppose $G = Spin(8)$. This group has the outer automorphism group $S_3 = \langle \rho, \sigma \rangle$, where $\rho^3 = \sigma^2 = (\rho\sigma)^2 = 1$, and two Majorana-Weyl real 8-dimensional representations that related to the 8-dimensional real vector representation by the action of S_3 . The group $Spin(8)$ contains the subgroup $G_1 \times (G_2 \ltimes G_3)$ that is isomorphic to the direct product of $U(1)_Y$ and $SU(2)_L \ltimes$

$U(3)_C$. Moreover, we always can choose these subgroups in the following manner:

- (i) $g^\phi \neq g$ for $1 \neq \phi \in S_3$ and $g \in G_1$,
- (ii) $g^\phi \neq g = g^\sigma$ for $1 \neq \phi \neq \sigma$ and $g \in G_2$,
- (iii) $g^\phi = g$ for all $\phi \in S_3$ and $g \in G_3$.

Just as for $Spin(8)$, the group $S_3 \ltimes Spin(8)$ cannot contain the SM group as a subgroup. There is nevertheless at least one possibility to break the symmetry of $S_3 \ltimes Spin(8)$ down to the SM group, namely, if we suppose that the subgroups G_2 and G_3 act in nonintersecting subspaces of \tilde{V} . Then we could in principle consider the symmetry breaking $\tilde{G} \rightarrow G_2 \times G_3$ instead of $\tilde{G} \rightarrow G_2 \ltimes G_3$ and hence assume that the symmetry is broken as

$$S_3 \ltimes Spin(8) \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y. \quad (4)$$

(In the final section we shall discuss briefly characteristic feature of the model.) Arguing as above, we see that in this case the gauge couplings of the SM group should be satisfy

$$g_3 = \sqrt{3}g_2 = \sqrt{6}g_1 \quad (5)$$

as $\mu = M_0$. Thus, as a result of the symmetry breaking (4) the gauge couplings and the normalization of the generators of SM group should be change.

3 Gauge couplings unification

The conditions (5) are valid in the $S_3 \ltimes Spin(8)$ limit. Now we need to study the regime $\mu < M_0$. The evolution of the SM gauge coupling constants in the one-loop approximation is controlled by the renormalization group equation

$$\frac{d\alpha_n^{-1}(\mu)}{d \ln \mu} = \frac{b_n}{6\pi}, \quad (6)$$

where $b_1 = -2n_1$, $b_2 = 22 - 2n_2$, $b_3 = 33 - 2n_3$, and $\alpha_n = g_n^2/4\pi$. (We have ignored the contribution coming from the Higgs scalar and higher-order effects.) It follows from (5) that the generators of SM group in the fundamental representation should be normalize by the condition $6n_3 = 2n_2 = n_1 = N_f$,

where N_f is the number of quark flavours. Expressing the low-energy couplings in terms of more familiar parameters, we can represent the solutions of Eq. (6) as

$$\alpha_s^{-1}(\mu) = \alpha_s^{-1}(M_0) - \frac{b_3}{6\pi} \ln \frac{M_0}{\mu}, \quad (7)$$

$$\alpha^{-1}(\mu) \sin^2 \theta_\mu = \alpha^{-1}(M_0) - \frac{b_2}{6\pi} \ln \frac{M_0}{\mu}, \quad (8)$$

$$\frac{3}{5} \alpha^{-1}(\mu) \cos^2 \theta_\mu = \alpha^{-1}(M_0) - \frac{b_1}{6\pi} \ln \frac{M_0}{\mu}. \quad (9)$$

Taking the linear combination $[12 \times \text{Eq. (7)} - 18 \times \text{Eq. (8)} + 7 \times \text{Eq. (9)}]$ and using the relations (5), we have

$$\sin^2 \theta_\mu = \frac{7}{37} + \frac{60}{111} \frac{\alpha(\mu)}{\alpha_s(\mu)}. \quad (10)$$

Obviously, Eq. (10) implies a non-trivial consistency condition among the gauge couplings. Taking the linear combination $[-78 \times \text{Eq. (7)} + 6 \times \text{Eq. (8)} + 10 \times \text{Eq. (9)}]$ and again using the relations (5), we have

$$\ln \frac{M_0}{\mu} = \frac{6\pi}{407} [\alpha^{-1}(\mu) - 13\alpha_s^{-1}(\mu)]. \quad (11)$$

This determines the unification scale M_0 . Also, combining Eqs. (10) and (11), we obtain

$$\sin^2 \theta_\mu = \frac{3}{13} - \frac{110\alpha(\mu)}{39\pi} \ln \frac{M_0}{\mu}. \quad (12)$$

Finally, it follows easily from Eqs. (7)–(9) that the running electroweak and strong gauge coupling constants satisfy

$$\alpha^{-1}(\mu) = \alpha^{-1}(M_0) - \frac{66 - 13N_f}{18\pi} \ln \frac{M_0}{\mu}, \quad (13)$$

$$\alpha_s^{-1}(\mu) = \alpha_s^{-1}(M_0) - \frac{99 - N_f}{18\pi} \ln \frac{M_0}{\mu}, \quad (14)$$

where the gauge couplings are connected by the relation $\alpha_s(M_0) = 13\alpha(M_0)$. Note that the choice of normalization of the generators essentially influences on the behaviour of the gauge couplings by changing its values in fixed points. So for example Eqs. (13) and (14) will differ from that obtained in the SM.

Therefore we must have a rule which permits to compare the gauge coupling constants in our (special) and the standard normalizations. We will extract this rule from the renormalization on-shell scheme.

The on-shell scheme [1, 2, 3, 4, 5, 6] (see also Ref. [7]) promotes the tree-level formula $\sin^2 \theta_W = 1 - M_W^2/M_Z^2$ to a definition of the renormalized $\sin^2 \theta_W$ to all orders in perturbation theory, i.e.,

$$\sin^2 \theta_W = \frac{\pi \alpha v^2}{M_W^2(1 - \Delta r)}, \quad (15)$$

where Δr summarizes the higher order terms. Here α is the fine structure constant, M_W is the mass of the charged gauge boson, and $v = (\sqrt{2}G_F)^{-1/2}$ is the vacuum expectation value. One finds $\Delta r = \Delta r_0 - \Delta r'$, where $\Delta r_0 = 1 - \alpha/\alpha(M_Z)$ is due to the running of α and $\Delta r'$ represents the top quark mass m_t and the Higgs boson mass M_H dependence. Using the formal expansion $(1 + \Delta r')^{-1} = 1 - \Delta r' + \dots$, we can rewrite the formula (15) in the form

$$\sin^2 \theta_W = \frac{\pi \alpha(M_Z) v^2}{M_W^2} \left(1 - \frac{\alpha(M_Z)}{\alpha} \Delta r' + \dots \right). \quad (16)$$

In the on-shell scheme the value of $\sin^2 \theta_W$ is independent of the normalization of the generators. We suppose that the value of $\alpha(\mu)$ in the fixed point $\mu = M_Z$ is the same for both the standard and special normalizations. Using this condition, we can now compare values of the running coupling constant $\alpha(\mu)$ in the two normalizations.

Following Ref. [8, 9, 10], we remove the (m_t, M_H) dependent term from Δr and write the renormalized $\sin^2 \theta_\mu$ (in the special normalizations) as

$$\sin^2 \theta_\mu = \frac{\pi \alpha(\mu') v^2}{M_W^2} \quad (17)$$

for $M_Z \leq \mu \leq \mu' \leq v$. Further, we suppose that the unification scale coincides with the electroweak scale (i.e., $M_0 = v$) and that the left-hand side of Eqs. (12) and (17) are equal as $\mu = M_0$. In this case

$$\frac{M_W}{M_0} = \sqrt{\frac{13\pi\alpha_0}{3}}, \quad (18)$$

where $\alpha_0 = \alpha(M_0)$. It follows from (10)–(14) and (18) that the three parameters $\alpha(M_Z)$, $\alpha_s(M_Z)$, and $\sin \theta_W$ of the SM are now determined in terms of one independent parameters α_0 . Thus, there are two predictions.

In conclusion, we show that the values of the coupling constants and the mass of the gauge bosons which are deduced from the SM are compatible with these predictions. Using $\alpha_0^{-1} = 127.726$ and $N_f = 6$ yield $\alpha^{-1}(M_Z) = 127.937$, $\alpha_s(M_Z) = 0.1221$, and $\sin^2 \theta_W = 0.2229$. With more careful treatment of two-order effects, one obtains $\alpha_s(M_Z) = 0.1210$. (Other parameters are changed unessentially.) These values are compatible with the SM predictions in Refs. [11, 12, 13, 14]. (For a recent review, see Ref. [15] and references therein.) This means, in particular that the value of $\alpha_s(M_Z)$ may also be chosen the same for both the standard and special normalizations. Using $M_0 = 246.2204$ GeV yield $M_W = 80.3841$ GeV and $M_Z = 91.1876$ GeV. This is also compatible with the SM predictions.

4 Discussion

We discuss briefly characteristic feature of the $S_3 \ltimes Spin(8)$ model. The group $Spin(8)$ occupies a special position among the simple Lie groups since only it has outer automorphism group S_3 . Namely this property of $Spin(8)$ permits to define the special normalization in a natural way and to get the gauge coupling constants unification. However, if the scale of such unification is of the same order as the scale of the electroweak symmetry breaking, then why the effects of the unification (such as effects due to additional gauge bosons) have not been seen at current colliders? The point is likely that the bigger symmetry can be broken in different ways: by introducing a higher-dimensional Higgs field, by a Green-Schwarz mechanism, by compactification on a non-trivial background manifold, or by compactification on an orbifold. But not always the symmetry breaking must be close to the Grand Unified scale. For example, in gauge-Higgs models, it is of the order of the electroweak scale.

The following question is whether the breaking of the discrete S_3 -symmetry has any physical meaning. To describe the breaking, we embedded the $Spin(8)$ -gauge theory in the model with larger global symmetry groups. The motivation for this is that whatever the high energy physics producing the spontaneous breaking of the gauge group, it is likely to possess a larger global symmetry than the gauged one. Moreover, we risk to suppose that due to the $S_3 \ltimes Spin(8)$ -symmetry breaking the quark and anti-quark fields acquire the colour degrees of freedom. But of course this is only a hypothesis.

Finally, we note that the $Spin(8)$ gauge group is closely related with the unique algebra of octonions. There have been many attempts over the

years to incorporate this algebra into physics. However, in most cases a clear physical framework has been lacking. The $S_3 \ltimes Spin(8)$ model contains many hints of an underlying octonionic structure, in particular in connection with the important role of triality in $Spin(8)$.

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